Zero Temperature Limits of Gibbs Equilibrium states for Countable Markov Shifts

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Abstract

We prove that, given a uniformly locally constant potential f on a countable state Markov shift and suitable conditions which guarantee the existence of the equilibrium states μ_{tf} for all t, the measures μ_{tf} converge in the weak star topology as t tends to infinity.

1 Introduction

Given a Markov shift Σ and a function $f : \Sigma \to \mathbb{R}$, we denote by μ_f the equilibrium state associated to f. In [3], Brémont proved the convergence as t tends to infinity of the sequence of equilibrium states μ_{tf} associated to a locally constant potential f on a finite topologically mixing Markov shift. In statistical mechanics this corresponds to proving that the equilibrium states of a system of particles at temperature $\frac{1}{t}$ converge to some ground state as t tends to infinity. Zero temperature limit laws are also of relevance to ergodic optimisation, because any limit point of μ_{tf} will be a maximising measure for the function f (maximising $\int f d\mu$). Indeed, the first study of zero temperature limits in the mathematical literature seems to be in [6], where it was used to find so called extremal measures.

Countable state Markov shifts are the subject of great interest and are models for many other systems, such as the Gauss map. In this article we extend the result of [3] to deal with countable state Markov shifts.

In [10] and [4], the results of Brémont were extended to show the convergence of equilibrium states μ_{tf+g} on finite state Markov shifts, where f is locally constant and g Hölder continuous. However, in [5] Chazottes and Hochman gave an example of a Hölder continuous potential f on a finite Markov shift for which the sequence μ_{tf} fails to converge.

In [7], Iommi proved the convergence of equilibrium states μ_{tf} for a locally constant potential f on a countable renewal type shift. In [8], Jenkinson, Mauldin and Urbański considered the equilibrium states μ_{tf} associated to Hölder continuous f on a countable alphabet with suitable conditions to ensure the existence of equilibrium states. They proved that such a sequence has at least one limit point, and asked whether the sequence converges.

If the sequence μ_{tf} does converge finding the limit can be useful, and in [8] zero temperature limits were described as the most "physically relevant" maximising measures. In the case that there is a unique maximising measure, or that among maximising measures there is one with greater entropy than all the others, the sequence μ_{tf} will converge to this measure (see [13]). In [4], a simple algortihm was given to find the zero temperature limit of μ_{tf} for f locally constant on a finite Markov shift. However in [1] an example was given to show that, in the case of Hölder continuous functions for which the zero temperature limit exists, the limit can behave strangely as f varies.

In this paper we consider uniformly locally constant potentials on a countable Markov shift under suitable conditions as given in [8] to ensure the existence of equilibrium measures μ_{tf} for all t. We prove that the equilibrium states μ_{tf} converge as t tends to infinity and that their limit can be found by first reducing to a finite Markov shift and then using the algorithm given in [4].

2 Set Up

We let $\mathcal{A}$ be a countable set and M be a matrix of zeroes and ones indexed by $\mathcal{A} \times \mathcal{A}$. We define the two sided topological Markov shift (Σ, σ) to be the set

$$\Sigma := \{\underline{x} \in \mathcal{A}^{\mathbb{Z}} : M_{x_i x_{i+1}} = 1 \ \forall i \in \mathbb{Z}\},$$

coupled with the shift map $\sigma : \Sigma \to \Sigma$, $(\sigma(\underline{x}))_i = (\underline{x})_{i+1}$. We further define the cylinder set $[x_0, \cdots, x_n]$ to be the set of sequences $\underline{y} = (y_i)_{i=-\infty}^{\infty}$ in Σ such that $y_0 \cdots y_n = x_0 \cdots x_n$.

Given a function $f : \Sigma \to \mathbb{R}$, we define the nth variation of f to be

$$var_n(f) := \sup\{|f(\underline{x}) - f(\underline{y})| : x_{-n} \cdots x_n = y_{-n} \cdots y_n\}.$$

We define a metric on Σ by

$$d(\underline{x}, \underline{y}) = 2^{-\sup\{N : x_{-N} \cdots x_N = y_{-N} \cdots y_N\}}.$$

f is Hölder continuous if there exist $c > 0$ and $\theta \in (0, 1)$ such that $var_n(f) < c\theta^n$ for all $n \in \mathbb{N}$. f has summable variation if $\sum_{n=1}^{\infty} var_n(f) < \infty$ and is called uniformly locally constant if there exists an n for which $var_n(f) = 0$.

Given a function $f : \Sigma \to \mathbb{R}$, we define the topological pressure of f:

$$P(f) := \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\sigma^n(\underline{x}) = \underline{x}} \exp(f^n(\underline{x})) 1_{[a]}(\underline{x}) \right)$$

where a is any element of $\mathcal{A}$ and $f^n(\underline{x}) := \sum_{k=0}^{n-1} f(\sigma^k(\underline{x}))$. This definition is independent of a and satisfies the variational principle:

$$P(f) = \sup\{h(\mu) + \int f d\mu < \infty : \mu \in \mathcal{M}, \int f d\mu > -\infty\},$$

where $h(\mu)$ denotes the metric entropy of μ and $\mathcal{M}$ the set of σ invariant Borel probability measures on Σ, see [14] for details. $\mathcal{M}$ is equipped with the weak* topology, we say $\mu_n \to \mu$ if and only if for every bounded continuous function $f : \Sigma \to \mathbb{R}$ we have $\int_\Sigma f d\mu_n \to \int_\Sigma f d\mu$, as in [2]. We call $\mu \in \mathcal{M}$ an equilibrium state for f if

$$P(f) = h(\mu) + \int f d\mu < \infty.$$

We call μ the Gibbs state associated to potential f if there exist constants $C_1, C_2 > 0$ such that for any $\underline{x} \in \Sigma$

$$C_1 \leq \frac{\mu[x_0, \cdots, x_{n-1}]}{\exp(f^n(\underline{x})) - nP(f))} \leq C_2$$

We define μ_f to be the Gibbs state associated to potential f. In the case that $\mathcal{A}$ is finite and f has summable variation, equilibrium states and shift invariant Gibbs states conincide. The situation for countable alphabets is more complicated, see [12] for details, but our conditions will ensure that for all $t \geq 2$, μ_{tf} is both the invariant Gibbs state and the equilibrium state for tf.

We call functions f and f' cohomologous, writing $f' \sim f$, if there exists a $\phi : \Sigma \to \mathbb{R}$ such that $f' = f + \phi - \phi \circ \sigma$. If there exists a constant c such that $f \sim f' + c$ then $\mu_f = \mu_{f'}$.

By recoding the shift and adding a coboundary if necessary, it is possible to assume that for a uniformly locally constant potential f we have $f(\underline{x}) = f(x_0 x_1)$.

We say a Markov shift Σ satisfies the big images and preimages property (BIP) if there exists a finite set $b_1, \cdots, b_N$ of elements of $\mathcal{A}$ such that for any $a \in \mathcal{A}$ there exist i, j with $M_{b_i a} M_{a b_j} = 1$.

It was shown in [11] and [15] that for a potential f with summable variation and finite pressure, BIP is necessary and sufficient for the existence of an invariant Gibbs measure μ_f associated to f. In our case that $f(\underline{x}) = f(x_0 x_1)$, summable variation corresponds to

$$\sup\{|f(\underline{x}) - f(\underline{y})| : x_0 = y_0\} < \infty.$$

BIP and summable variation coupled with finite topological pressure imply that

$$\sum_{i \in \mathcal{A}} \exp(\sup f|_{[i]}) < \infty.$$

Then for all $t \geq 2$ we have

$$\sum_{i \in \mathcal{A}} \sup(tf|_{[i]}) \exp(\sup tf|_{[i]}) < \infty,$$

(see [13], Lemma 3.1) which ensures that μ_{tf} is also an equilibrium state. We now have enough conditions to ensure the existence of μ_{tf} for all t and to state our theorem.

Theorem 2.1. *Let Σ be a Markov shift satisfying BIP and let $f : \Sigma \to \mathbb{R}$ be uniformly locally constant with summable variation and finite topological pressure. Then the equilibrium measures μ_{tf} exist for all $t \geq 2$ and converge in the weak* topology as t tends to infinity.*

It is known that in the case of a finite alphabet Markov shift and locally constant potential the zero temperature limit exists. Our method will be to relate μ_{tf} to the equilibrium states ν_{tf} of f on some finite subshift $\Sigma' \subset \Sigma$ and argue that, for any bounded continuous $g : \Sigma \to \mathbb{R}$, $\mu_{tf}(g) - \nu_{tf}(g|_{\Sigma'}) \to 0$ as $t \to \infty$, thus allowing us to use the convergence of the ν_{tf} on the finite subshift to imply the convergence of of the μ_{tf} on the countable subshift.

3 Recasting the Question

In this section we recast the question as one about the convergence of ratios of certain sums. It was proved in [9] that, given any pair (Σ, f) for which f has an equilibrium state μ_f, there exists at least one measure μ for which

$$\int f d\mu = \alpha(f) := \sup \left\{ \int f dm : m \in \mathcal{M} \right\}.$$

Such a measure is called a maximising measure and the set of maximising measures is denoted $\mathcal{M}_{max}(f)$. It was further proved in [9] that there exists some $f' \sim f$ with $f'(\underline{x}) = f'(x_0 x_1)$ and $f' \leq \alpha(f)$[1]. For ease of computation we replace f with $f' - \alpha(f)$, without affecting the equilibrium states of f. We now have that $\alpha(tf) = 0$ and $tf \leq 0$ for all $t \in \mathbb{R}$.

The conditions $\sum_{i \in \mathcal{A}} \exp(\sup f|_{[i]}) < \infty$ and $\sup f = 0$ imply that there exists a finite subset $I = \{i_1 \cdots i_k\}$ of $\mathcal{A}$ upon which $\sup f|_{[i]} = 0$, and that there exists a constant $d > 0$ such that $\sup f|_{[i]} \leq -d$ for all i not in $\mathcal{A}$. All maximising measures have their support contained in $\cup_{i \in I}[i]$. It was proved in [8] that all the accumulation points of the sequence of equilibrium states μ_{tf} are in the set $\mathcal{M}_{max}(f)$ and hence that $\lim_{t \to \infty} \mu_{tf}([i_1] \cup \cdots \cup [i_k]) = 1$. Finally, it was proved in [13] that any limit point of μ_{tf} has entropy equal to $h := \sup\{h(m) : m \in \mathcal{M}_{max}(f)\}$.

We split I into components, letting i and j be in the same component if there exists an ergodic maximising measure giving positive measure to both $[i]$ and $[j]$. There exists at most one measure m with $h(m) = h$ on each component, and thus any limit point of

[1]In fact, in [9] the authors worked with functions of summable variation, but if f depends only on 2 coordinates then the ϕ_λ of Lemma 4.2 of their paper depends only on one coordinate, and this property is inherited by the ϕ_1 of section 5. Composing with the shift we get a coboundary depending only on 2 coordinates. We thank Oliver Jenkinson for explaining this point to us.

μ_{tf} must be a convex combination of these maximising measures of maximal entropy. This reduces the problem of showing that μ_{tf} converges to the problem of showing that $\lim_{t\to\infty} \frac{\mu_{tf}[a]}{\mu_{tf}[b]}$ exists (possibly being infinite) for all $a, b \in I$.

Given a word $\alpha = \alpha_0 \cdots \alpha_n$ we define $\mu(\alpha) = \mu[\alpha_0 \cdots \alpha_n]$ and $f(\alpha) := \sum_{k=0}^{n-1} f(\sigma^k(\alpha_0 \cdots \alpha_n))$. Then the Gibbs inequality guarantees that, for a closed loop γ based at γ_0, we have

$$\mu_{tf}[\gamma] = \mu_{tf}[\gamma_0] \exp((tf - P(tf))(\gamma)).$$

Since μ_{tf} is a probability measure, the above equation can be phrased as saying points in $[a]$ follow path γ with probability $\exp((tf - P(tf))(\gamma))$.

Now let $a, b \in I$. For each t, $\mu_{tf}[a], \mu_{tf}[b] > 0$ since μ_{tf} is a Gibbs measure and hence fully supported. Then the set

$$A := \{\underline{x} \in \Sigma : x_n = a \text{ for infinitely many positive and negative } n\}$$

has $\mu_{tf}(A) = 1$, because A is a σ-invariant set of positive measure and σ is ergodic. So $\mu_{tf}[b] = \mu_{tf}([b] \cap A)$.

We enumerate $(\gamma_i)_{i=1}^{\infty}$ the set of loops $\{\gamma = x_1 \cdots x_n, x_j = a \text{ iff } j \in \{1, n\}\}$ and let $l(\gamma_i)$ be the number of symbols in the loop γ_i. Then A is partitioned by the set $\{\sigma^k[\gamma_i] : i \in \mathbb{N}, 1 \le k \le l(\gamma_i)\}$, and so $[b] \cap A$ is partitioned by the set

$$\left\{\sigma^k[\gamma_i] : i \in \mathbb{N}, 1 \le k \le l(\gamma_i), \sigma^k[\gamma_i] \in [b]\right\}$$

We let $N(b, \gamma_i)$ denote the number of occurrences of the symbol b in loop γ_i. Then

$$\begin{aligned} \mu_{tf}[b] &= \sum_{i=1}^{\infty} \sum_{k=1}^{l(\gamma_i)} \mu_{tf}(\sigma^k[\gamma_i]).\chi_{[b]}(\sigma^k[\gamma_i]) \\ &= \sum_{i=1}^{\infty} \mu_{tf}[\gamma_i] N(b, \gamma_i) \\ &= \sum_{i=1}^{\infty} \mu_{tf}[a] \exp((tf - P_{tf})(\gamma_i)) N(b, \gamma_i), \end{aligned}$$

and hence

$$\frac{\mu_{tf}[b]}{\mu_{tf}[a]} = \sum_{i=1}^{\infty} \exp((tf - P_{tf})(\gamma_i)) N(b, \gamma_i).$$

Now $(\gamma_i)_{i=1}^{\infty}$ is the set of all loops from a to a which have no occurrence of a in the middle. Loops γ_i which do not pass through b do not effect the above equation and we disregard them. All other loops γ_i can be split into three pieces, a path from a to b with no intermediate occurence of a or b, $N(b, \gamma_i) - 1$ loops from b to b with no intermediate occurence of a or b, and a path from b to a with no intermediate occurence of a or b.

For $i, j \in \{a, b\}$ we denote by $\{\alpha : i \to j\}$ the set of paths $\alpha = \alpha_1, \cdots, \alpha_m$ with $\alpha_1 = i, \alpha_m = j, \alpha_i \notin \{a, b\}$ for $i \in \{2, \cdots, m-1\}$. Then

$$
\begin{aligned}
\frac{\mu_{tf}[b]}{\mu_{tf}[a]} &= \sum_{n=0}^{\infty}(n+1)\left(\sum_{\alpha:a\to b}\exp((tf-P_{tf})(\alpha))\right)\left(\sum_{\alpha:b\to b}\exp((tf-P_{tf})(\alpha))\right)^n \\
&\times \left(\sum_{\alpha:b\to a}\exp((tf-P_{tf})(\alpha))\right).
\end{aligned}
$$

Each of these summations is a sum of positive terms. The finiteness of $\frac{\mu_{tf}[b]}{\mu_{tf}[a]}$ guarantees the finiteness of each summation, and so the sums must converge. This guarantees that the following is well defined, for $i,j \in \{a,b\}$ we define

$$
p^t_{ij} = \sum_{\alpha:i\to j\in\Sigma}\exp((tf-P_{tf})(\alpha)).
$$

From above we have

$$
\begin{aligned}
\frac{\mu_{tf}[b]}{\mu_{tf}[a]} &= p^t_{ab}p^t_{ba}\left(\sum_{n=0}^{\infty}(n+1)(p^t_{bb})^n\right) \\
&= \frac{p^t_{ab}p^t_{ba}}{(1-p^t_{bb})^2}.
\end{aligned}
$$

Similarly

$$
\frac{\mu_{tf}[a]}{\mu_{tf}[b]} = \frac{p^t_{ab}p^t_{ba}}{(1-p^t_{aa})^2},
$$

and so dividing through we see that

$$
\frac{\mu_{tf}[b]}{\mu_{tf}[a]} = \frac{1-p^t_{aa}}{1-p^t_{bb}}.
$$

Thus we have reduced the problem of showing that the μ_{tf} converge to showing that the ratio $\frac{1-p^t_{aa}}{1-p^t_{bb}}$ converges for each $a,b \in I$. In the next section we define a relevant finite Markov shift and use the above form for the measure to compare equilibrium states on the finite shift with those on the countable shift.

4 A related finite Markov shift

The following three definitions define a finite Markov shift upon which equilibrium states for tf approximate μ_{tf}..

Definition 4.1. *Given Σ, f and $c \in \mathbb{R}$ we define Σ_c to be the subshift of finite type Σ restricted to alphabet $\{i \in \mathcal{A}$ such that $\sup_{\underline{x}\in i}\{f(\underline{x})\} \geq c\}$.*

Definition 4.2. *We let $c > 0$ and $N \in \mathbb{N}$ be constants such that for each $a,b \in I$:*

1. *There exists a loop γ in Σ passing through a and b with $f(\gamma) > -c$ and $l(\gamma) \leq N$.*
2. *If there exists a loop γ in Σ passing through a and b and avoiding some set $I' \subset I$, then there exists such a loop with $f(\gamma) > -c$ and $l(\gamma) \leq N$.*
3. *The connected component of Σ_{-c} containing all of I is topologically mixing.*

Such constants exist because I is a finite set and Σ satisfies BIP.

Definition 4.3. *We define Σ' to be the path connected component of Σ_{-7c} with vertices including all of the elements of I.*

(Σ', σ) is topologically mixing. We denote by ν_{tf} the equilibrium measures of $tf|_{\Sigma'}$. The term $P(tf)$ is now ambiguous, we let P_{tf} denote the topological pressure of the potential tf on Σ and Q_{tf} denote the pressure of tf on Σ'. $P_{tf} > Q_{tf}$.

Defining $q^t_{ij} = \sum_{\alpha: i \to j \in \Sigma'} \exp(tf - Q_{tf})(\alpha))$, the analysis of the previous section yields

$$\frac{\nu_{tf}[b]}{\nu_{tf}[a]} = \frac{1 - q^t_{aa}}{1 - q^t_{bb}}.$$

Now the convergence of $\dfrac{\nu_{tf}[b]}{\nu_{tf}[a]}$, which is the main result of [3], ensures the convergence of $\dfrac{1 - q^t_{aa}}{1 - q^t_{bb}}$. We will use this to prove the convergence of $\dfrac{1 - p^t_{aa}}{1 - p^t_{bb}}$ and hence of $\dfrac{\mu_{tf}[b]}{\mu_{tf}[a]}$.

4.1 Convergence

In order to prove asymptotic convergence of $1 - p^t_{aa}$ to $1 - q^t_{aa}$ we need lower bounds on $1 - p^t_{aa}$ and $1 - q^t_{aa}$. The probability, with respect to μ_{tf}, that a path from a returns to a eventually is one. We split this into p^t_{aa}, the probability that a path from a goes to a without passing through b, and $p^t_{ab} p^t_{ba} \sum_{n=0}^{\infty} (p^t_{bb})^n$, the probability that a path returns to a passing through b at least once. So

$$p^t_{aa} + p^t_{ab} p^t_{ba} \sum_{n=0}^{\infty} (p^t_{bb})^n = p^t_{aa} + \frac{p^t_{ab} p^t_{ba}}{1 - p^t_{bb}} = 1$$

We recall that by the definition of c there exists some path γ from a to a passing through b with $f(\gamma) \geq -c$ and $l(\gamma) \leq N$. This path is included in the summation $p^t_{ab} p^t_{ba} \sum_{n=0}^{\infty} p^t_{bb}$, so

$$1 - p^t_{aa} = p^t_{ab} p^t_{ba} \sum_{n=0}^{\infty} (p^t_{bb})^n \geq \exp(-tc - NP_{tf})$$

The same arguments work for q^t_{ij}.

We introduce a third term, r^t_{ij}, for summation over paths between i and j, and recall the earlier definitions for comparison.

$$\begin{aligned}
p^t_{ij} &= \sum_{\alpha: i\to j\in\Sigma} \exp((tf - P_{tf})(\alpha)) \\
q^t_{ij} &= \sum_{\alpha: i\to j\in\Sigma'} \exp(tf - Q_{tf})(\alpha)) \\
r^t_{ij} &= \sum_{\alpha: i\to j\in\Sigma'} \exp(tf - P_{tf})(\alpha)).
\end{aligned}$$

We have $r^t_{ij} \leq p^t_{ij}$ and $r^t_{ij} \leq q^t_{ij}$. We use the following technical lemma, the proof of which is deferred to the final section.

Lemma 4.1. *There exists a $K \in \mathbb{R}$ such that for all t, $p^t_{aa} \leq r^t_{aa} + K(\exp(-3ct))$, $p^t_{bb} \leq r^t_{bb} + K(\exp(-3ct))$ and $p^t_{ab}p^t_{ba} \leq r^t_{ab}r^t_{ba} + K(\exp(-3ct))$.*

Now let $a(t) \asymp b(t)$ mean $\lim_{t\to\infty} \dfrac{a(t)}{b(t)} = 1$. The following two simple lemmas prove that $\dfrac{1-p^t_{aa}}{1-p^t_{bb}} \asymp \dfrac{1-q^t_{aa}}{1-q^t_{bb}}$, completing the proof of Theorem 2.1.

Lemma 4.2. $1 - p^t_{aa} \asymp 1 - r^t_{aa}$, $1 - p^t_{bb} \asymp 1 - r^t_{bb}$

Proof: We have that $1 - p^t_{aa} \geq \exp(-ct - NP_{tf})$. Then

$$1 \leq \frac{1-r^t_{aa}}{1-p^t_{aa}} = 1 + \frac{p^t_{aa} - r^t_{aa}}{1-p^t_{aa}} \leq 1 + \frac{K(\exp(-3ct)}{\exp(-ct - NP_{tf})} \to 1$$

giving $1 - p^t_{aa} \asymp 1 - r^t_{aa}$. Identical arguments work for p^t_{bb}.

Lemma 4.3. $1 - r^t_{aa} \asymp 1 - q^t_{aa}$, $1 - r^t_{bb} \asymp 1 - q^t_{bb}$.

Proof: Since $P_{tf} > Q_{tf}$ we have immediately that $1 - r^t_{aa} > 1 - q^t_{aa}$. We consider the other direction.

Now using Lemma 4.1 we have

$$\begin{aligned}
r^t_{aa} + \frac{r^t_{ab}r^t_{ba}}{1-r^t_{bb}} &\geq p^t_{aa} - K(\exp(-3ct)) + \frac{p^t_{ab}p^t_{ba} - K(\exp(-3ct))}{1 - p^t_{bb} + K(\exp(-3ct))} \\
&= p^t_{aa}(1 - \frac{K(\exp(-3ct))}{p^t_{aa}}) + \frac{p^t_{ab}p^t_{ba}}{1-p^t_{bb}} \left(\frac{1 - \frac{K(\exp(-3ct)}{p^t_{ab}p^t_{ba}}}{1 + \frac{K(\exp(-3ct))}{1-p^t_{bb}}} \right) \\
&\geq \left(p^t_{aa} + \frac{p^t_{ab}p^t_{ba}}{1-p^t_{bb}} \right) \left(\frac{1 - K(\exp((3c-c)t))}{1 + K(\exp((-3c+c)t))} \right) \\
&= 1 - o(\exp(-ct))
\end{aligned}$$

So substituting $q^t_{aa} + \dfrac{q^t_{ab}q^t_{ba}}{1-q^t_{bb}} = 1$ gives

$$r^t_{aa} + \frac{r^t_{ab}r^t_{ba}}{1-r^t_{bb}} \geq q^t_{aa} + \frac{q^t_{ab}q^t_{ba}}{1-q^t_{bb}} - o(\exp(-ct)),$$

and so

$$1 - r^t_{aa} \;\leq\; 1 - q^t_{aa} - \frac{q^t_{ab}q^t_{ba}}{1-q^t_{bb}} + \frac{r^t_{ab}r^t_{ba}}{1-r^t_{bb}} + o(\exp(-ct)).$$

Then, since $\frac{r^t_{ab}r^t_{ba}}{1-r^t_{bb}} \leq \frac{q^t_{ab}q^t_{ba}}{1-q^t_{bb}}$, we have

$$1 - r^t_{aa} \leq 1 - q^t_{aa} + o(\exp(-ct)).$$

Finally using $1 - q^t_{aa} \geq \exp(-ct - NP_{tf})$ we conclude that

$$1 - r^t_{aa} \asymp 1 - q^t_{aa}.$$

as required.

Then

$$\frac{\mu_{tf}[b]}{\mu_{tf}[a]} = \frac{1-p^t_{aa}}{1-p^t_{bb}} \asymp \frac{1-r^t_{aa}}{1-r^t_{bb}} \asymp \frac{1-q^t_{aa}}{1-q^t_{bb}} = \frac{\nu_{tf}[b]}{\nu_{tf}[a]}$$

which converges, and so the limit $\lim_{t\to\infty} \frac{\mu_{tf}[b]}{\mu_{tf}[a]}$ exists, giving us finally that $\lim_{t\to\infty}\mu_{tf}$ exists and proving Theorem 2.1.

The exact value of $\lim_{t\to\infty}\nu_{tf}$ is given by an algorithm in [4] which terminates after finitely many steps. Then since $\lim_{t\to\infty}\mu_{tf}$ is the same, we have that the zero temperature limit for the countable case can be given by reducing Σ to Σ' and then following the same algorithm.

5 Proof of Technical Lemma

Lemma 5.1. *There exists a $K \in \mathbb{R}$ such that $p^t_{aa} \leq r^t_{aa} + K(\exp(-3ct))$, $p^t_{bb} \leq r^t_{bb} + K(\exp(-3ct))$ and $p^t_{ab}p^t_{ba} \leq r^t_{ab}r^t_{ba} + K(\exp(-3ct))$ for all t.*

Proof: We prove the inequality for p^t_{aa}, the same argument works for p^t_{bb} and a similar one for $p^t_{ab}p^t_{ba}$.

For any path $\alpha : a \to a$ we let $n(\alpha)$ be the number of occurrences of elements of I in α. We define the set X^n_{aa} to be the set of possible sequences $a = i_1, i_2, \cdots, i_n = a$ of elements of I in paths $\alpha : a \to a$ with $n(\alpha) = n$. Then, writing $\alpha : i_k \hookrightarrow i_{k+1}$ for paths α from i_k to i_{k+1} not passing through any other element of I, we have

$$\begin{aligned} p^t_{aa} &= \sum_{n=2}^{\infty} \sum_{i_1,\cdots,i_n \in X^n_{aa}} \prod_{k=1}^{n-1} \left(\sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma} \exp((tf - P_{tf})(\alpha)) \right) \text{ and} \\ r^t_{aa} &= \sum_{n=2}^{\infty} \sum_{i_1,\cdots,i_n \in X^n_{aa}} \prod_{k=1}^{n-1} \left(\sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma'} \exp((tf - P_{tf})(\alpha)) \right). \end{aligned}$$

We define

$$p_{aa}^t(n) = \sum_{i_1,\cdots,i_n \in X_{aa}^n} \prod_{k=1}^{n-1} \left(\sum_{\alpha: i_k \hookrightarrow i_{k+1} \in \Sigma} \exp((tf - P_{tf})(\alpha)) \right),$$

this is the summation p_{aa}^t restricted to those paths α with $n(\alpha) = n$. We let $r_{aa}^t(n)$ be defined similarly. We define

$$\epsilon(n) = \frac{p_{aa}^t(n)}{r_{aa}^t(n)}.$$

Then

$$0 \leq p_{aa}^t - r_{aa}^t = \sum_{n=1}^{\infty} p_{aa}^t(n) \left(1 - \frac{1}{\epsilon(n)}\right)$$

We claim that there exists a K_2 such that for $n \in \{r|I|, r|I|+1, \cdots (r+1)|I|-1\}$,

1. $p_{aa}^t(n) \leq (1 - \exp(-ct - NP_{tf}))^r$
2. $\epsilon(n) \leq (1 + K_2 \exp(-5ct))^r$ for $r \geq 1$
3. $\epsilon(n) \leq (1 + K_2 \exp(-5ct))$ for $n \in \{1, \cdots, |I|-1\}$

Then

$$\begin{aligned}
p_{aa}^t - r_{aa}^t &= \sum_{n=1}^{\infty} p_{aa}^t(n) \left(1 - \frac{1}{\epsilon(n)}\right) \\
&\leq \sum_{n=1}^{\infty} p_{aa}^t(n) \left(\epsilon(n) - 1\right) \\
&\leq |I| \sum_{r=0}^{\infty} (1 - \exp(-ct - NP_{tf}))^r \left((1 + K_2 \exp(-5ct))^r - 1\right) \\
&= \frac{|I|}{1 - (1 - \exp(-ct - NP_{tf}))(1 + K_2 \exp(-5ct))} - \frac{|I|}{\exp(-ct - NP_{tf})} \\
&\leq \frac{|I|}{\exp(-ct - NP_{tf}) - K_2 \exp(-5ct))} - \frac{|I|}{\exp(-ct - NP_{tf})} \\
&\leq K \exp(-3ct)
\end{aligned}$$

It remains only to prove claims 1, 2 and 3.

Proof of claim 1:

To find an upper bound on $p_{aa}^t(n)$, we infact find an upper bound on $\sum_{j=n}^{\infty} p_{aa}^t(j)$. By the definition of c, there exists for any i_k a closed loop γ based at i_k passing through a, avoiding b, and with $f(\gamma) \geq -c, l(\gamma) \leq N$. We can remove any subloops from γ without decreasing $f(\gamma) \geq -c$, since $f \leq 0$. Then γ contains a path from i_k to a passing through at most $|I|$ elements of I. Elements of $[i_k]$ follow path γ with (μ_{tf}) probability $\exp((tf - P_{tf})(\gamma)) \geq \exp(-ct - NP_{tf})$. Then in particular, the probability that an element of $[i_k]$ passes through at most $|I|$ elements of I before returning to $[a]$ is greater than or equal to $\exp(-ct - NP_{tf})$. So $\sum_{k=1}^{\infty} p_{aa}^t(k) \leq 1$ and

$$\sum_{k=m+|I|}^{\infty} p_{aa}^t(k) \leq (1 - \exp(-tc - NP_{tf})) \sum_{k=m}^{\infty} p_{aa}^t(k),$$

giving that for $n \in \{r|I|, r|I|+1, \cdots (r+1)|I|-1\}$,

$$p_{aa}^t(n) \leq \sum_{k=r|I|}^{\infty} p_{aa}^t(k) \leq (1 - \exp(-tc - NP_{tf}))^r$$

Proof of claims 2 and 3: We recall that P_{tf} decreases to h, the maximum entropy of any maximising measure, and that $d > 0$ is such that $\sup f|_{[i]} \leq -d$ for all $i \notin I$. We let T be such that $P_{Tf} < h + d$, and have that for all $t > T$,

$$-d < P_{(t+1)f} - P_{tf} < 0.$$

We consider $(tf - P_{tf})$ evaluated along a path $\alpha = \alpha_0 \cdots \alpha_m : i_k \hookrightarrow i_{k+1} \in \Sigma \setminus \Sigma', m \geq 2$. We have $f(\alpha_0\alpha_1) \leq 0$ and $f(\alpha_n\alpha_{n+1}) < -d$ for $1 \leq n < m$, because α_n is not an element of I for $1 \leq n < m$. Furthermore, since $\alpha \in \Sigma \setminus \Sigma'$, there exists at least one n for which $f(\alpha_n\alpha_{n+1}) < -7c$. So

$$\begin{aligned}
((t+1)f - P_{(t+1)f})(\alpha) - (tf - P_{tf})(\alpha) &= (f - P_{(t+1)f} + P_{tf})(\alpha) \\
&= m(P_{tf} - P_{(t+1)f}) + f(\alpha) \\
&\leq md - 7c - (m-1)(d) \\
&= -7c + d \\
&\leq -6c
\end{aligned}$$

for $t > T$. Then defining

$$K_1 := \exp(6cT) \sup_{i_k, i_{k+1}} \sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma \setminus \Sigma'} \exp((Tf - P_{Tf})(\alpha))$$

we have that for any choices of i_k, i_{k+1} and for any $t > T$,

$$\sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma \setminus \Sigma'} \exp((tf - P_{tf})(\alpha)) \leq K_1 \exp(-6ct)$$

Now by the definition of c there exists some path $\beta : i_k \hookrightarrow i_{k+1} \in \Sigma'$ with $f(\beta) \geq -c$. So for $t > T$,

$$\sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma \setminus \Sigma'} \exp((tf - P_{tf})(\alpha)) \leq K_1 \exp(-5ct) \left(\sum_{\alpha : i_k \hookrightarrow i_{k+1} \in \Sigma'} \exp((tf - P_{tf})(\alpha)) \right)$$

giving

$$\sum_{\alpha:i_k\hookrightarrow i_{k+1}\in\Sigma} \exp((tf-P_{tf})(\alpha)) \le (1+K_1\exp(-5ct))\left(\sum_{\alpha:i_k\hookrightarrow i_{k+1}\in\Sigma'} \exp((tf-P_{tf})(\alpha))\right).$$

So for any $n \in \{r|I|, r|I|+1, \cdots (r+1)|I|-1\}$ we have

$$\begin{aligned}\frac{p^t_{aa}(n)}{r^t_{aa}(n)} &\le (1+K_1\exp(-5ct))^n \\ &\le \left((1+K_1\exp(-5ct))^{2|I|}\right)^r,\end{aligned}$$

for $r \ge 1$, and using $K_2 = 2^{2|I|}K_1$ we have

$$\frac{p^t_{aa}(n)}{r^t_{aa}(n)} \le (1+K_2\exp(-5ct))^r$$

for $r \ge 1$. For $n \in \{1, \cdots, |I|-1\}$ we have $\frac{p^t_{aa}(n)}{r^t_{aa}(n)} \le (1+K_2\exp(-5ct))$. This completes the proof.

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